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Stability of plane Couette–Poiseuille flow

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The stability of a two-dimensional Couette–Poiseuille flow is investigated. The primary unidirectional flow is between two infinite parallel plates, one of which moves relative to the other. The results for the case of Poiseuille flow agree with Lin's results and all flows for which the plate velocity exceeds 70 % of the maximum velocity of the Poiseuille component of flow are found to be stable for all finite Reynolds numbers. Results of intermediate cases are also obtained.

1. Introduction

The stability of plane Couette flow and plane Poiseuille flow has been the subject of many investigations in recent years. The question of whether plane Couette flow is or is not stable to all infinitesimal disturbances and for all Reynolds numbers, however large, is still not definitely settled. Recently, Deardorff (1963) has shown by a numerical computation that this flow is stable for Reynolds numbers less than 5720 (based on channel width and boundary speed) and probably stable for all Reynolds numbers. As to plane Poiseuille flow, Lin (1945) has verified Heisenberg's (1924) earlier conclusion that plane Poiseuille flow is unstable, and has determined the critical Reynolds number to be 10,600 (based on channel width and maximum velocity). His results have now been generally accepted. But the stability of combined plane Couette and Poiseuille flows has not been investigated so far, and this investigation is to furnish the missing information.

It can be expected that a combination of plane Couette and Poiseuille flow will yield instability only at large Reynolds numbers although one does not know *a priori* that a superposition of Couette flow on Poiseuille flow will cause the flow to be more or less stable. If instability at large Reynolds number is assumed, asymptotic solutions of the governing Orr–Sommerfeld equation involving this parameter are appropriate. These asymptotic solutions have singularities at critical points where the wave velocity of the disturbance is equal to the velocity of the primary flow. These singularities, not inherent in the governing equation but introduced entirely by the method of solution, present difficulties which must be attended to with great care.

This study, which was also proposed (but unsolved) by Lin in 1945, is a variation of the problem discussed in this paper. The difference between this study and previous works is that in this investigation both relative motion and a pressure

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gradient are allowed. Difficulty arises in the numerical solution of the secular equation because there may be two critical points when the entire velocity profile is used, requiring asymptotic expansions about each point. Lin avoided this difficulty by utilizing the symmetry of the primary velocity profile which he treated.

2. Governing differential system

For steady two-dimensional flow of an incompressible, viscous fluid flowing between two parallel plates (see figure 1) the governing Navier–Stokes equation can be integrated to yield the velocity distribution of the primary flow

$$\bar{u}(Y) = 2W(Y^2 - Yd) + (U_2/d)\,Y,$$

which in non-dimensional form is

$$U(y) = -4(y^2 - Ay), \tag{1}$$

where $U = \bar{u}/W, \quad y = Y/d, \quad A = (U_2/4W) + 1,$

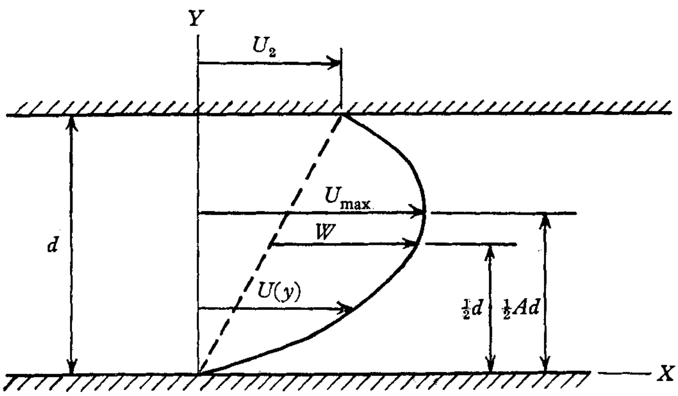

FIGURE 1. Primary velocity distribution.

with W denoting the maximum velocity of the Poiseuille component of the flow, d the distance between plates, and U_2 the speed of the upper plate. If A equals unity, Poiseuille flow results and as A approaches infinity the flow tends toward Couette flow.

Let the stream function for the small disturbance be

$$\psi = \phi(y)\, e^{i\alpha(x-ct)}$$

and after the Navier–Stokes equations are linearized and the pressure eliminated by cross-differentiation the Orr–Sommerfeld equation

$$(U-c)(\phi'' - \alpha^2\phi) - U''\phi = -(i/\alpha R)(\phi^{\text{iv}} - 2\alpha^2\phi'' + \alpha^4\phi) \tag{2}$$

results, where α is the non-dimensional wave number, $R = Wd/\nu$ is the Reynolds number, ν is the kinematic viscosity, and $c = c_r + ic_i$ is the complex wave-speed.

The boundary conditions are

$$\phi(0) = \phi'(0) = \phi(1) = \phi'(1) = 0.$$

The eigen-value problem thus formed requires that

$$c_i(\alpha, R) = 0$$

if no growth or decay of the disturbance is allowed. The minimum Reynolds number from the neutral-stability curve representing the above relationship is the critical Reynolds number sought after in this study.

3. Asymptotic solutions

There exist four independent solutions of equation (2) which will be denoted by $\phi_1(y)$, $\phi_2(y)$, $\phi_3(y)$, $\phi_4(y)$. Heisenberg (1924) has given two asymptotic methods, each of which yields two particular solutions of equation (2). The first of these methods is to expand $\phi(y)$ in powers of $(\alpha R)^{-1}$, namely

$$\phi(y) = \phi^{(0)}(y) + (\alpha R)^{-1}\phi^{(1)}(y) + (\alpha R)^{-2}\phi^{(2)}(y) + \dots$$

and substitute into equation (2). The term of zero order in $(\alpha R)^{-1}$ yields the inviscid equation

$$(U-c)(\phi^{(0)''} - \alpha^2\phi^{(0)}) - U''\phi^{(0)} = 0. \tag{3}$$

Heisenberg obtained two solutions as convergent series of α^2 in the forms

$$\left.\begin{aligned} \phi_1^{(0)}(y) &= (U-c)\,[h_0(y) + \alpha^2 h_2(y) + \dots], \\ \phi_2^{(0)}(y) &= (U-c)\,[k_1(y) + \alpha^2 k_3(y) + \dots], \end{aligned}\right\} \tag{4}$$

where

$$\left.\begin{aligned} h_0(y) &= 1, \\ k_1(y) &= \int_0^y (U-c)^{-2}\,dy, \\ h_{2n+2}(y) &= \int_0^y (U-c)^{-2}\,dy \int_0^y (U-c)^2 h_{2n}(y)\,dy, \\ k_{2n+3}(y) &= \int_0^y (U-c)^{-2}\,dy \int_0^y (U-c)^2 k_{2n+1}(y)\,dy. \end{aligned}\right\} \tag{5}$$

These integrals are complex. The imaginary part is obtained by integrating around the critical point where $U = c$. Heisenberg obtained two other solutions in the form

$$\phi(y) = f(y)\exp[\pm\sqrt{(\alpha R)}\,g(y)].$$

Expanding $f(y)$ in powers of $(\alpha R)^{-\frac{1}{2}}$ and substituting into equation (2) yields, to a first order approximation,

$$\phi_3(y) = (U-c)^{-\frac{5}{4}}\exp\left[-\int_{y_c}^{y} \sqrt{\{i\alpha R(U-c)\}}\,dy\right], \tag{6}$$

$$\phi_4(y) = (U-c)^{-\frac{5}{4}}\exp\left[\int_{y_c}^{y} \sqrt{\{i\alpha R(U-c)\}}\,dy\right]. \tag{7}$$

These solutions are not valid in the neighbourhood of the critical points where $U = c$. Thus if a boundary occurs close to a critical point (6) and (7) should not be used to evaluate ϕ_3 and ϕ_4 at that boundary. Solutions must be found which are valid near the critical points and which approach the solutions (6) and (7) away from the critical points.

For a parabolic distribution there are, at most, two critical points. Following the procedure of Lin, expand $\phi(y)$ in powers of $(\alpha R)^{-\frac{1}{3}}$ and make a change of scale such that

$$\phi(\zeta) = \chi^{(0)}(\zeta) + (\alpha R)^{-\frac{1}{3}}\chi^{(1)}(\zeta) + \dots, \tag{8}$$

where, near the first singular point

$$\zeta = (y-y_c)(U_c'\alpha R)^{\frac{1}{3}}, \tag{9}$$

and near the second singular point

$$\zeta = (y-y_c)(-U_c'\alpha R)^{\frac{1}{3}}. \tag{10}$$

Substituting into equation (2) yields

$$\frac{d^4\chi^{(0)}}{d\zeta^4} \pm i\zeta\frac{d^2\chi^{(0)}}{d\zeta^2} = 0, \tag{11}$$

since $\chi^{(0)''} \gg \chi^{(0)}$

and near the critical points $U-c = U_c'(y-y_c)$.

The negative sign is used near the first singular point and the positive sign near the second. This method yields four solutions near each singular point. It can be shown that the solutions near the first singular point

$$\chi_3 = \int_\infty^\zeta d\zeta \int_\infty^\zeta \zeta^{\frac{1}{2}} H_{\frac{1}{3}}^{(1)}[\tfrac{2}{3}(i\zeta)^{\frac{3}{2}}]\, d\zeta, \tag{12}$$

$$\chi_4 = \int_{-\infty}^\zeta d\zeta \int_{-\infty}^\zeta \zeta^{\frac{1}{2}} H_{\frac{1}{3}}^{(2)}[\tfrac{2}{3}(i\zeta)^{\frac{3}{2}}]\, d\zeta, \tag{13}$$

and near the second singular point

$$\Omega_3 = \int_\infty^\zeta d\zeta \int_\infty^\zeta (-\zeta)^{\frac{1}{2}} H_{\frac{1}{3}}^{(2)}[\tfrac{2}{3}(-i\zeta)^{\frac{3}{2}}]\, d\zeta, \tag{14}$$

$$\Omega_4 = \int_{-\infty}^\zeta d\zeta \int_{-\infty}^\zeta (-\zeta)^{\frac{1}{2}} H_{\frac{1}{3}}^{(1)}[\tfrac{2}{3}(-i\zeta)^{\frac{3}{2}}]\, d\zeta, \tag{15}$$

approach the solutions ϕ_3 and ϕ_4, given in equations (6) and (7), as one moves away from the singular point.

If instability occurs at a wave-number larger than 0·8 the terms in equations (4) may not decrease rapidly enough for a simple solution. With a parabolic velocity it may be easier to solve (3) by the Frobenius method. By using this method the two solutions found in a region containing the first singular point are

$$\phi_1 = \sum_{n=1}^{\infty} a_n(y-y_c)^n, \tag{16}$$

$$\phi_1 = \phi_2 \ln(y-y_c) + \sum_{n=1}^{\infty} b_n(y-y_c)^n, \tag{17}$$

where $-\pi < \arg(y-y_c) < 0$. The a_n's and b_n's are given by

$$a_1 = 1, \quad a_2 = -1/B, \quad a_3 = \tfrac{1}{6}\alpha^2,$$

$$a_n = (1/n(n-1)B)[n(n-3)a_{n-1} + a^2Ba_{n-2} - \alpha^2 a_{n-3}],$$

$$b_0 = -\tfrac{1}{2}B, \quad b_1 = 1, \quad b_2 = (1/B) - \tfrac{1}{4}\alpha^2 B,$$

$$b_n = \frac{1}{n(n-1)B}[n(n-3)b_{n-1} + B\alpha^2 b_{n-2} - \alpha^2 b_{n-3} + (2n-3)a_{n-1} - B(2n-1)a_n],$$

where $B = \sqrt{(A^2-c)}$.

In a region containing the second singular point the solutions are

$$\psi_1 = \sum_{n=1}^{\infty} c_n(y-y_c)^n, \tag{18}$$

$$\psi_2 = \sum_{n=0}^{\infty} d_n(y-y_c)^n + \psi_1[\ln(y-y_c) - i\pi], \tag{19}$$

where $0 < \arg(y-y_c) < \pi$. The c_n's and d_n's are given by

$$c_1 = 1, \quad c_2 = 1/B, \quad c_3 = \tfrac{1}{6}\alpha^2,$$

$$c_n = [-1/n(n-3)\,c_{n-1} + \alpha^2 B c_{n-2} + {}^2c_{n-3}]/n(n-1)\,B,$$

$$d_0 = \tfrac{1}{2}B, \quad d_1 = 1, \quad d_2 = -(1/B) + \tfrac{1}{4}\alpha^2 B,$$

$$d_n = [-n(n-3)\,d_{n-1} + \alpha^2 B d_{n-2} + \alpha^2 d_{n-3} - (2n-3)\,c_{n-1} - B(2n-1)\,c_n]/n(n-1)\,B.$$

ϕ_1 is only valid up to the second critical point and hence, for the case when two critical points exist between the boundaries, the solutions must be matched in a region of common validity. In this region

$$\phi_1 = A_1\psi_1 + B_1\psi_2, \tag{20}$$

$$\phi_2 = A_2\psi_1 + B_2\psi_2. \tag{21}$$

From the form of the governing equation it is sufficient to match the function and its first derivative only at one point. Denoting this point by the subscript '0'

$$\phi_{10} = A_1\psi_{10} + B_1\psi_{20},$$

$$\phi_{10}' = A_1\psi_{10}' + B_1\psi_{20}',$$

$$\phi_{20} = A_2\psi_{10} + B_2\psi_{20},$$

$$\phi_{20}' = A_2\psi_{10}' + B_2\psi_{20}',$$

from which A_1, B_1, A_2, and B_2 are calculated.

Now four solutions have been found near each critical point. Near the first they are given by equations (12), (13), (16), and (17), or by (12), (13), and (4). Near the second critical point they are given by (14), (15), (20), and (21), or by (14), (15), and (4). Away from the critical point ($|\zeta| > 6$) solution for ϕ_3 and ϕ_4 given by (6) and (7) should be used.

4. The secular eqation

Denoting the lower boundary by the superscript '1' and the upper boundary with the subscript '2' the boundary conditions demand that the secular equation

$$\begin{vmatrix} \phi_{11} & \phi_{21} & \phi_{31} & \phi_{41} \\ \phi_{11}' & \phi_{21}' & \phi_{31}' & \phi_{41}' \\ \phi_{12} & \phi_{22} & \phi_{32} & \phi_{42} \\ \phi_{12}' & \phi_{22}' & \phi_{32}' & \phi_{42}' \end{vmatrix} = 0 \tag{22}$$

be satisfied. From an order-of-magnitude analysis this equation can be simplified to

$$\frac{f_1}{f_3}-\frac{\phi_{31}}{\phi'_{31}}+\frac{\phi_{42}f_4}{\phi'_{42}f_3}\left(\frac{\phi_{31}}{\phi'_{31}}-\frac{f_2}{f_4}\right) = 0, \tag{23}$$

where

$$f_1 = \phi_{11}\phi_{22}-\phi_{21}\phi_{12},$$
$$f_2 = \phi_{11}\phi'_{22}-\phi_{21}\phi'_{12},$$
$$f_3 = \phi'_{11}\phi_{22}-\phi'_{21}\phi_{12},$$
$$f_4 = \phi'_{11}\phi'_{22}-\phi'_{21}\phi'_{12}.$$

The expressions in equation (23) vary depending on the location of the critical points with respect to the boundaries and may change as α becomes small. Each possibility will now be studied.

(i) *Critical points near the boundaries, α not small* (see figure 2)

Since two critical points are in the region of interest, at the lower boundary ϕ_1, ϕ_2, ϕ_3, and ϕ_4 are given by equations (12), (13), (16), and (17) and at the upper boundary by (14), (15), (20), and (21). No simplifications can be made.

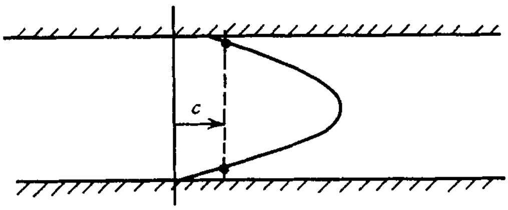

FIGURE 2. Two critical points in the region of interest.

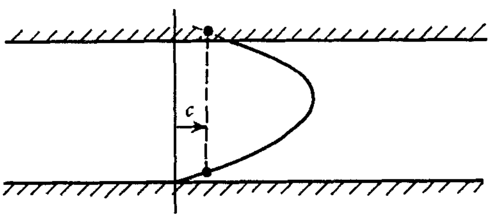

FIGURE 3. One critical point in the region of interest.

(ii) *A single critical point near the lower boundary, α not small* (figure 3)

Here only one critical point is in the region of interest. Thus ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are given by equations (12), (13), (16), and (17) at the lower boundary and (14), (15), (16), and (17) at the upper boundary. Again no simplifications can be made. It should be pointed out that this case requires approximately 15 terms in the series solutions for ϕ_1 and ϕ_2.

(iii) *Critical points near the boundaries, small α*

For this possibility expressions for ϕ_3 and ϕ_4 would be the same as those used in cases 1 and 2; but the expressions obtained for ϕ_1 and ϕ_2 by Heisenberg, equations (4), could now be used. This would be necessary if the primary velocity distribution were not parabolic, since the method of Frobenius cannot be used for a non-parabolic distribution. From equation (4) it can be seen that

$$f_1 = -c\phi_{22},$$
$$f_2 = -c\phi'_{22},$$
$$f_3 = U'_1\phi_{22} + 1/c\phi_{12},$$
$$f_4 = U'_1\phi'_{22} + 1/c\phi'_{12},$$

where
$$\phi_{12} = (U_2-c)\,[h_0(1)+\alpha^2 h_2(1)],$$
$$\phi_{22} = (U_2-c)\,[k_1(1)+\alpha^2 k_3(1)],$$
$$\phi'_{12} = (U_2-c)\,[\alpha^2 h'_2(1)+\alpha^4 h'_4(1)] + (U_2-c)^{-1}\,U'_2\,\phi_{12},$$
$$\phi'_{22} = (U_2-c)\,[\alpha^2 k'_3(1)] + (U_2-c)^{-1}\,U'_2\,\phi_{22}.$$

The series in the expressions for ϕ_1 and ϕ_2 converge quite rapidly since the h's, k's and their derivatives decrease with increasing number of integrations. Only three integrations can be performed without the use of extensive numerical integrations by a computer. Hence the range of application of this method is limited.

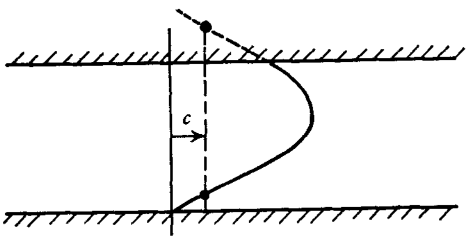

FIGURE 4. One critical point in the region of interest.

(iv) *Second critical point far from the boundary*
(see figure 4)

If the second critical point is far from the boundary then Heisenberg's solutions for ϕ_3 and ϕ_4 must be used near the top boundary. From equation (7) it is easily verified that

$$\phi_{42}/\phi'_{42} = (U_2-c)/[\exp(\tfrac{1}{4}i\pi)\,\surd\{\alpha R(U_2-c)^3\} - \tfrac{5}{4}U'_2]. \tag{24}$$

If $[\alpha R(U_2-c)^3]^{\frac{1}{2}} \gg 1$, this can be simplified to

$$\phi_{42}/\phi'_{42} = \exp[-\tfrac{1}{4}i\pi]/\surd\{\alpha R(U_2-c)\}.$$

If the first critical point is too far from the lower boundary Heisenberg's solutions would have to be used there. This is not expected in this study.

With the appropriate solutions, equation (23) must now be solved. Setting c_i equal to zero yields two non-linear algebraic equations (the real and imaginary parts)

$$\left.\begin{aligned} f(c_r,\alpha,R) &= 0,\\ g(c_r,\alpha,R) &= 0,\end{aligned}\right\} \tag{25}$$

in which c_r, α, and R are the unknowns. For a given c_r it is then possible to solve these equations simultaneously for α and R using a trial and error method. Newton's method (or similar methods) does not give solutions which converge to the roots of the equations because of the large gradients involved. An initial guess at the solutions must be very close, too close to make these methods worthwhile. The method used here was to fix α and plot the two functions on the left-hand side of (25) against R, then vary α by a small amount and again plot the two functions of R. When the two functions have the same zero the desired solution results. The calculations were done on the IBM 7090 computer at the University of Michigan.

5. Results and conclusions

The results show that the superposing of a Couette flow on a Poiseuille flow has a definite stabilizing effect, the effect being so pronounced that a speed of the boundary equal to 10 % of the maximum Poiseuille velocity increases the critical Reynolds number from 10,800 to 25,000 or an increase of 236 %. On the other

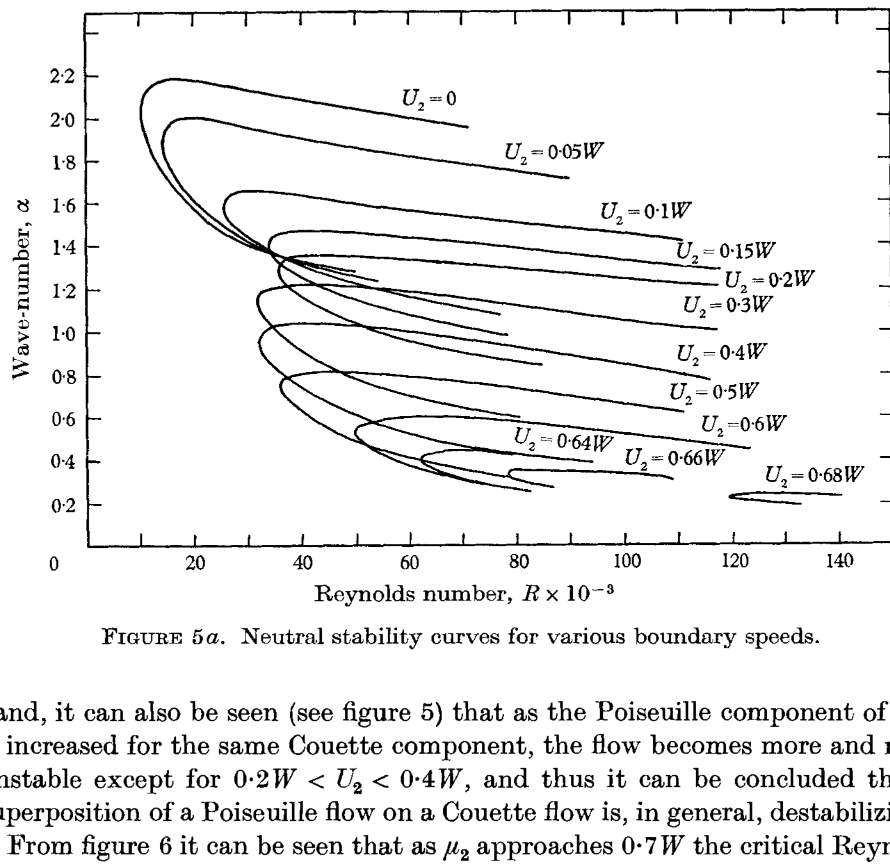

FIGURE 5*a*. Neutral stability curves for various boundary speeds.

hand, it can also be seen (see figure 5) that as the Poiseuille component of flow is increased for the same Couette component, the flow becomes more and more unstable except for $0{\cdot}2W < U_2 < 0{\cdot}4W$, and thus it can be concluded that a superposition of a Poiseuille flow on a Couette flow is, in general, destabilizing.

From figure 6 it can be seen that as μ_2 approaches $0{\cdot}7W$ the critical Reynolds number approaches infinity and the critical wave number approaches zero. Hence it can be concluded from this study that *all flows for which* $U_2 > 0{\cdot}7W$ *are stable* to infinitesimal disturbances for all Reynolds numbers. Critical Reynolds numbers for which $U_2 < 0{\cdot}7W$ can be found with the analysis given in this paper and are plotted in figure 6. The results give further support to the claim that pure Couette flow is always stable.

Figure 6 also shows a small range, $0{\cdot}4W > U_2 > 0{\cdot}2W$, for which the flow becomes more unstable with increased Couette flow. It is perhaps significant that this effect occurs when U_2 is approximately equal to the wave speed and that the point where $U_2 = c$ appears to be the inflexion point.

Increased Couette component of the flow has a destabilizing effect on waves with relatively small wave-numbers and a stabilizing effect on waves with relatively large wave-numbers. In all cases an increased Couette component decreases the critical wave-number. This effect is shown in figure 5.

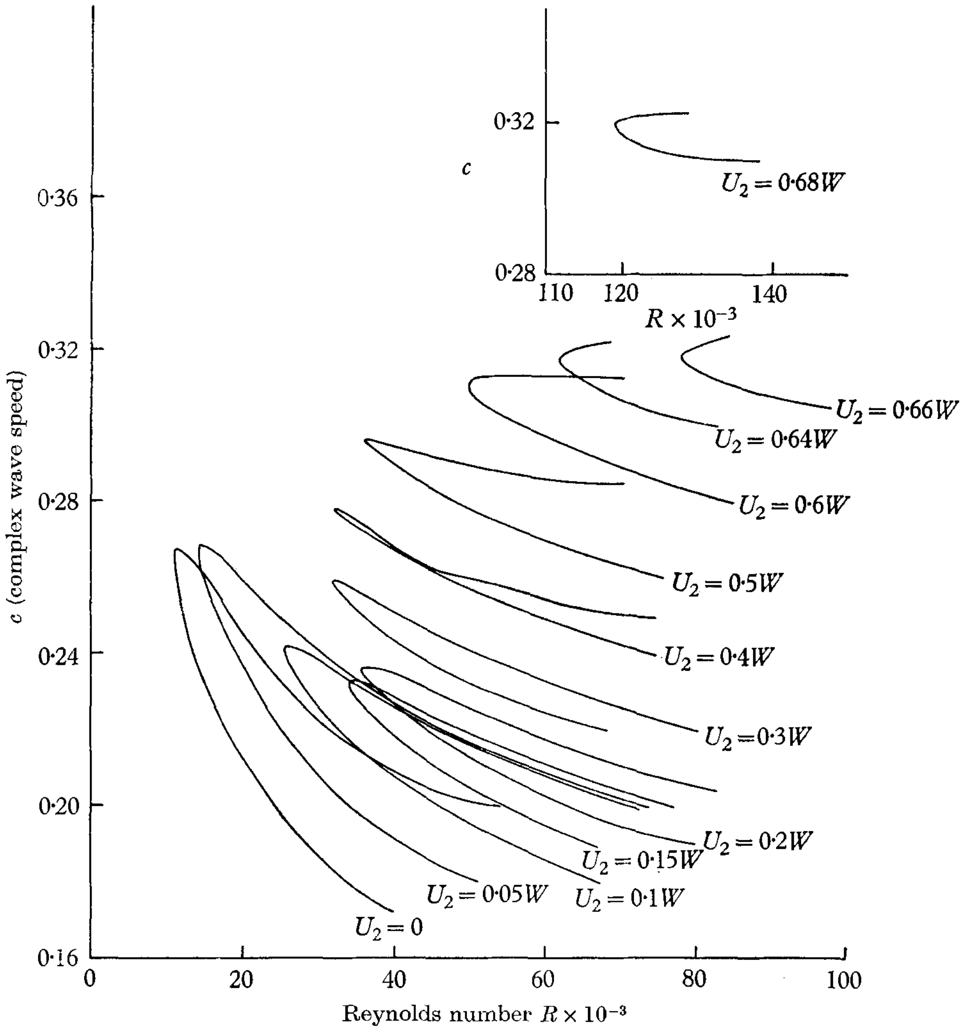

FIGURE 5*b*. Variation of complex wave speed with Reynolds number for various boundary speeds.

The curves for $U_2 = 0$ in figure 5 represent Poiseuille flow and agree very well with the results obtained by C. C. Lin. In this connexion one should remember that Lin used one-half of the channel width as a non-dimensionalizing parameter and hence the wave-numbers and Reynolds numbers in this study are double those obtained by Lin. There is slight disagreement for $\alpha > 1·6$ since Lin used series solutions in powers of α^2 for ϕ_1 and ϕ_2 which do not converge rapidly enough.

Values of c, α and R for values of U_2 from 0 to $0{\cdot}2W$ were found by using equations (16), (17), and (12) at the lower boundary and (20), (21), and (15) at the upper boundary as in case (i) in §4; and for values of U_2 from $0{\cdot}3$ to $0{\cdot}6W$ by using equations (16), (17), and (12) at the lower boundary and (16), (17), and (15)

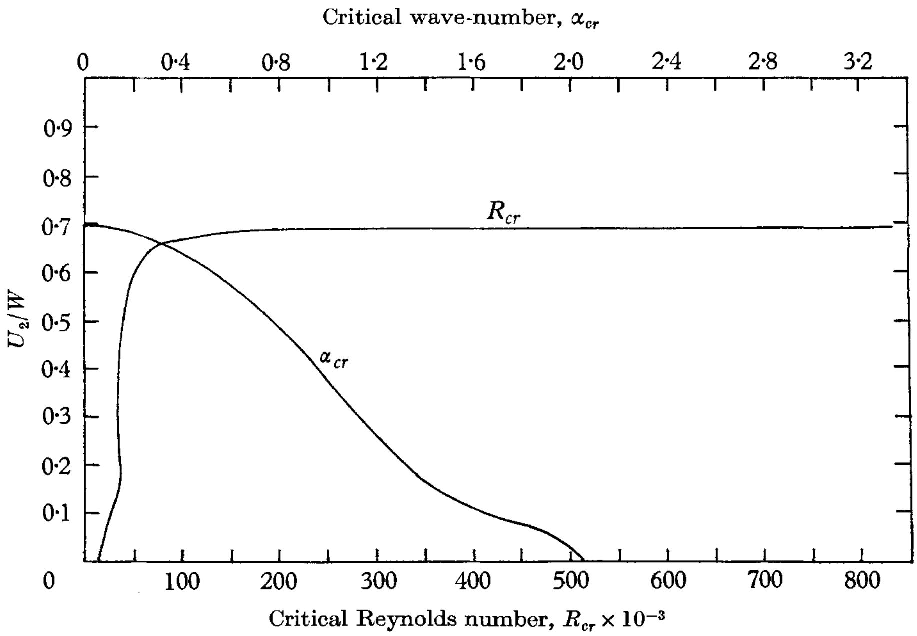

FIGURE 6. U_2/W *vs.* critical Reynolds number and critical wave-number.

c	α	R
0·310	0·180	174,000
0·315	0·197	147,000
0·317	0·196	143,000
0·318	0·194	141,000
0·319	0·190	142,000
0·320	0·184	143,000
0·321	0·177	147,000
0·322	0·166	153,000

TABLE 1. Neutral stability for $U_2 = 0{\cdot}685W$

at the upper boundary as in case (ii) in §4. Case (iv), in which Heisenberg's solution for ϕ_4 is used at the upper boundary, is illustrated in figure 5 for

$$U_2 = 0{\cdot}6W,\ 0{\cdot}66W \quad \text{and} \quad 0{\cdot}68W$$

and in tables 1–3.

Tables giving the values of c, α and R for the neutral stability curves in figure 5 are available on request from the Editor of the *Journal.*

Asymptotic behaviour (for very large R) of the neutral stability curves has not been studied in this report since the stability of the flow is of primary interest.

The results of this study were checked by finding the neutral stability curves for negative U_2. This can be thought of as simply reversing the directions of y and superposing a uniform flow, which should not affect the neutral stability curve. This was indeed found to be the case, that is, identical stability curves were found for $\pm U_2$.

c	α	R
0·310	0·119	266,000
0·313	0·139	216,000
0·315	0·147	198,000
0·317	0·148	190,000
0·318	0·146	189,000
0·319	0·143	190,000
0·320	0·137	194,000
0·321	0·128	203,000
0·322	0·117	220,000

TABLE 2. Neutral stability for $U_2 = 0{\cdot}69W$

c	α	R
0·315	0·066	443,000
0·317	0·0732	387,000
0·3175	0·0730	385,000
0·318	0·072	386,000
0·319	0·068	405,000
0·320	0·059	459,000

TABLE 3. Neutral stability for $U_2 = 0{\cdot}695W$

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